Benchmark Example No. 26

Response of a SDOF System to Harmonic Excitation

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VERiFiCATiON
BE26 Response of a SDOF System to Harmonic Excitation

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Front Cover

Volkstheater, Munich   Photo: Florian Schreiber

Overview	
Element Type(s):	SPRI, DAMP
Analysis Type(s):	DYN
Procedure(s):	TSTP
Topic(s):	
Module(s):	DYNA
Input file(s):	harmonic_damped.dat, harmonic_undamped.dat

1 Problem Description

This problem consists of an elastic SDOF system undergoing forced vibration, as shown in Fig. 1. The response of an undamped and damped system is determined and compared to the reference solution.

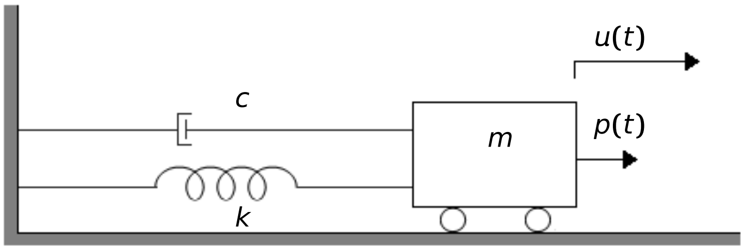


Figure 1: Problem Description

2 Reference Solution

A harmonic force is $p(t) = p_o \sin \omega_p t$, where p_o is the amplitude value of the force and its frequency ω_p is called the exciting frequency. The differential equation governing the forced harmonic vibration of a damped system is given by [1] [2]:

$$m\,\ddot{u} + c\,\dot{u} + k\,u = p_o \sin \omega_p t \tag{1}$$

$$m\,\ddot{u} + k\,u = p_o \sin \omega_p t \tag{2}$$

For undamped systems it simplifies to Eq. 2. Subjected also to initial conditions, $u(0)$ and $u(\dot{0})$, the total solution to Eq. 2 is:

$$u(t) = \underbrace{u(0)\cos \omega_n t + \left[\frac{\dot{u}(0)}{\omega_n} - \frac{p_o}{k}\frac{\omega_p/\omega_n}{1-(\omega_p/\omega_n)^2}\right]\sin \omega_n t}_{transient} + \underbrace{\frac{p_o}{k}\frac{1}{1-(\omega_p/\omega_n)^2}\sin \omega_p t}_{steadystate} \tag{3}$$

Eq. 3 shows, that u(t) contains two distinct vibration components, first the term $\sin \omega_p t$ gives a vibration at the exciting frequency and second the terms $\sin \omega_n t$ and $\cos \omega_n t$ give a vibration at the natural frequency of the system. The first term is the steady state vibration, corresponding to the applied force and the latter is the transient vibration, depending on the initial conditions. It exists even if the initial conditions vanish, in which case it becomes

$$u(t) = \frac{p_o}{k} \frac{1}{1-(\omega_p/\omega_n)^2} \left[sin\, \omega_p t - \frac{\omega_p}{\omega_n} sin\, \omega_n t \right] \tag{4}$$

For the case of a damped SDOF system, the total solution is given by

$$u(t) = \underbrace{e^{-\xi \omega_n t} [A\, cos\, \omega_D t + B\, sin\, \omega_D t]}_{transient} + \underbrace{C\, sin\, \omega_p t + D\, cos\, \omega_p t}_{steadystate} \tag{5}$$

The coefficients C and D are determined from the particular solution of the differential equation of motion (Eq. 1), whereas A and B are determined in terms of the initial conditions. For the special case of zero initial conditions, the coefficients are given by

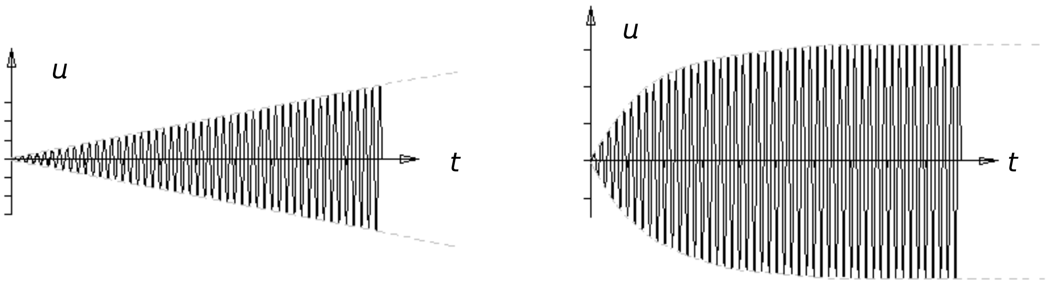
$$C = \frac{p_o}{k} \frac{1-(\omega_p/\omega_n)^2}{[1-(\omega_p/\omega_n)^2]^2 + [2\xi(\omega_p/\omega_n)]^2} \tag{6}$$

$$D = \frac{p_o}{k} \frac{-2\xi(\omega_p/\omega_n)}{[1-(\omega_p/\omega_n)^2]^2 + [2\xi(\omega_p/\omega_n)]^2} \tag{7}$$

$$A = -D \tag{8}$$

$$B = \frac{A\,\xi - C\,(\omega_p/\omega_n)}{\sqrt{1-\xi^2}} \tag{9}$$

For the special case where the exciting frequency equals the natural frequency of the SDOF system, we observe the resonant response. For the undamped system, the steady state response amplitude tends towards infinity as we approach unity and the peak values build up linearly, as shown in Fig. 2. For the damped case though, they build up in accordance to $(u_{st}/2\xi)e^{-\xi\omega_n t}$ and towards a steady state level, as shown in Fig. 2. The static deformation $u_{st} = p_o/k$, corresponds to the displacement which would be produced by the load p_o if applied statically, and serves as a measure of amplitude.



Undamped system Damped system

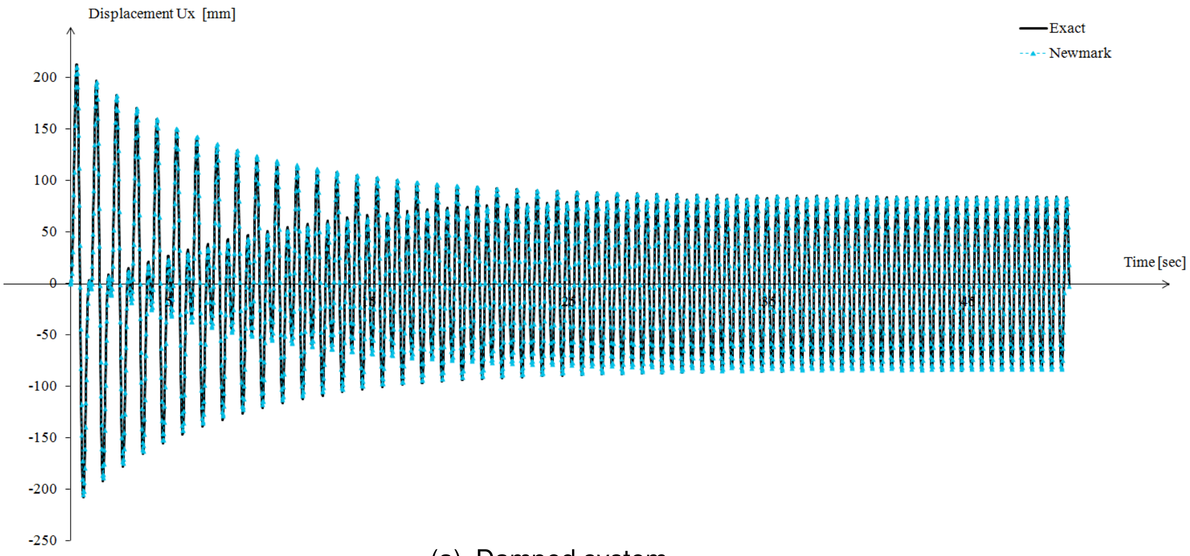
Figure 2: Response to Resonant Loading for at-rest initial conditions

3 Model and Results

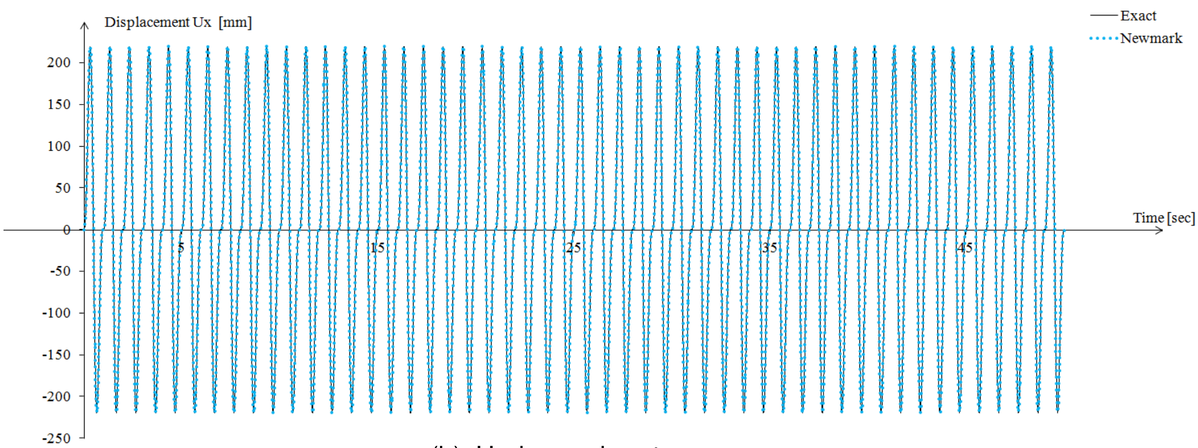
The properties of the model are defined in Table 1. The system is excited by a harmonic sinusoidal force and undergoes a forced vibration with zero initial conditions. The cases of the elastic damped and undamped SDOF system with a frequency ratio ω_p/ω_n = 2 are examined and their responses are compared to the exact solutions presented in Section 2. The resonance response is also examined for both systems, as shown in Fig. 4.

Table 1: Model Properties

Model Properties	Excitation Properties
$m = 1\,t$	$u(0) = 0$
$k = 4\pi^2\,kN/m$	$u(\dot{0}) = 0$
$T = 1\,sec$	$p_0 = 10\,kN$
$\xi = 2\,\%$	$\omega_p = 2\,\omega_n$

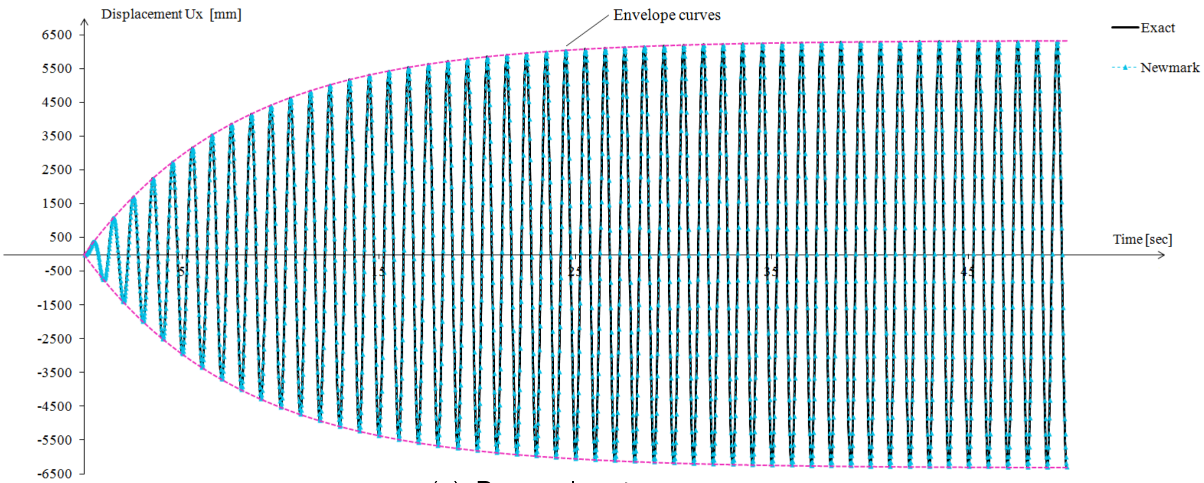


(a) Damped system

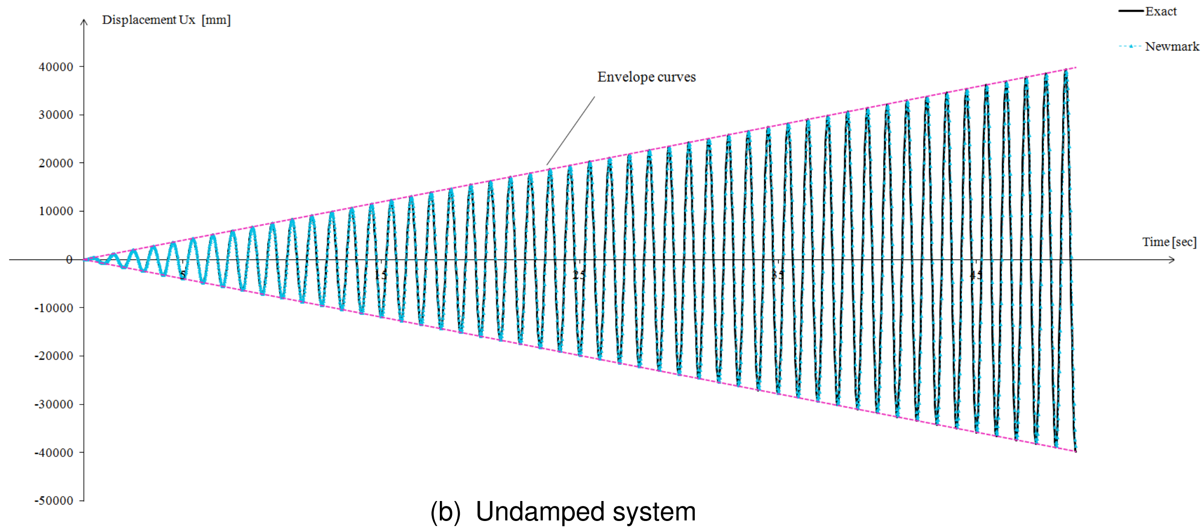


(b) Undamped system

Figure 3: Response to Harmonic Loading for at-rest initial conditions and ratio $\omega_p/\omega_n = 2$: (a) $\xi = 2\%$, (b) $\xi = 0$



(a) Damped system



(b) Undamped system

Figure 4: Response to Resonant Loading ($\omega_p/\omega_n = 1$) for at-rest initial conditions: (a) $\xi = 2\%$, (b) $\xi = 0$

4 Conclusion

The purpose of this example is to test the calculation of the response of a dynamic system in terms of a harmonic loading function. It has been shown that the behaviour of the system is captured adequately.

5 Literature

[1] R. W. Clough and J. Penzien. *Dynamics of Structures*. 3rd. Computers & Structures, Inc., 2003.

[2] A. K. Chopra. *Dynamics of Structures: Theory and Applications to Earthquake Engineering*. Prentice Hall, 1995.