Benchmark Example No. 6

Warping Torsion Bar

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VERiFiCATiON
BE6 Warping Torsion Bar

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Front Cover

Arnulfsteg, Munich   Photo: Hans Gössing

1 Problem Description

A cantilever I-bar is fixed at both ends, as shown in Fig. 1, and subjected to a uniformly distributed torque m_T [1]. Determine the angle of twist ϕ at the midspan.
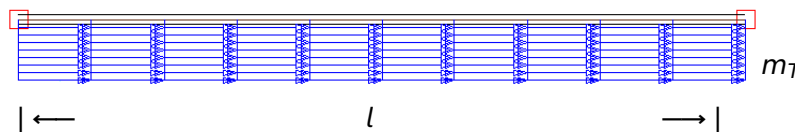

Figure 1: An I-bar with Uniformly Distributed Torque

2 Reference Solution

In mechanics, torsion is the twisting of a structure due to an applied torque. There are two types of torsion: St. Venant torsion and warping torsion. St. Venant torsion exists always when an element is twisted, whereas the warping torsion occurs additionally under specific conditions. The warping of a section depends on the section geometry which means that there exist warping-free, such as circular, and warping-restrained sections. St. Venant torsion is based on the assumption that either the cross-section is warping-free or that the warping is not constrained. If at least one of these conditions is not met then the warping torsion appears [2].

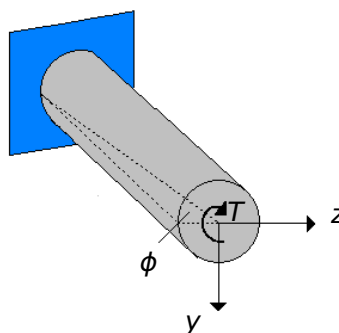

Figure 2: Circular Shaft

A member undergoing torsion will rotate about its shear center through an angle of ϕ. Consider a circular shaft that is attached to a fixed support at one end. If a torque T is applied to the other end, the shaft

will twist, with its free end rotating through an angle ϕ called the angle of twist [3]:

$$\phi = \frac{TL}{GI_T},$$ (1)

where G is the shear modulus and I_T the torsional moment of inertia. For a circular shaft subjected to torsion, each cross-section rotates along the shaft as a solid rigid slab (warping-free cross-section). The torsional moment resisted by the cross-section is:

$$T = GI_T \frac{d\phi}{dx},$$ (2)

For most cross-sections, e.g. non-circular, this rotation of the cross-section is accompanied by warping. Then the total torsional moment resisted by the cross-section becomes the sum of the pure torsion and warping torsion [4]. The stresses induced on the member is then classified into three categories: torsional shear stress, warping shear stress and warping normal stress. For example, when a bar of an I-cross-section is subjected to torsion, then the flanges of the cross-section experience bending in the flange planes. This means that torsion induces bending about the strong axis of the flanges. When the tendency for the cross-section to warp freely is prevented or restrained, it causes stresses to develop. The torque that the cross-section carries by bending is:

$$T = EC_M \frac{d^3\phi}{dx^3}$$ (3)

where EC_M, is the warping torsion stiffness. Furthermore, in warping torsion theory the bimoment is defined as an auxiliary quantity. The objective is to introduce a degree of freedom for beam elements that represents the torque due to restrained warping. The bimoment M_ω is defined as:

$$M_\omega = EC_M \frac{d^2\phi}{dx^2}$$ (4)

It should be noted, that the bimoment itself is not measurable, however it serves as a convenient parameter to quantify this prevention of warping.
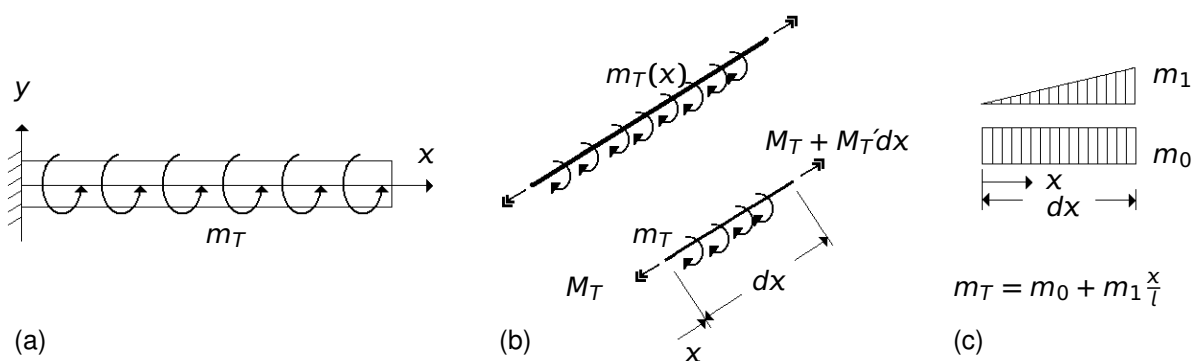

Figure 3: The Warping Torsion Problem

Fig. 3 (a) shows the warping torsion problem of a bar subjected to a distributed external torque. The differential equation governing the warping torsion problem, for a constant cross-section, becomes [5]:

$$EC_M \frac{d^4\phi}{dx^4} - GI_T \frac{d^2\phi}{dx^2} = m_T,$$ (5)

where m_T the distributed torque along the bar. The natural boundary conditions are:

$$M_\omega = EC_M \frac{d^2\phi}{dx^2}, \qquad z = 0 \text{ or } l$$ (6)

and

$$-EC_M \frac{d^3\phi}{dx^3} + GI_T \frac{d\phi}{dx} = M_T, \qquad z = 0 \text{ or } l$$ (7)

where M_T is the concentrated end torque and M_ω the bimoment. Introducing λ, the so called decay factor, in the above equation, and a simplified notation for the derivatives of ϕ, we obtain:

$$\phi'''' - \lambda^2 \phi'' = \frac{m_T}{EC_M}.$$ (8)

The solution of the warping torsion equation depends on the type of the torsional load and the kinematic boundary conditions, especially the amount of prevention of the warping. The complete solution system of Eq. 8, for the load type given in Fig. 3 (c), is thus:

$$\phi = \frac{C_1}{\lambda^2} \sinh \lambda x + \frac{C_2}{\lambda^2} \cosh \lambda x + C_3 x + C_4 - \frac{1}{2GI_T}(m_0 + \frac{1}{3}m_1 \frac{x}{l})x^2$$ (9)

$$\phi' = \frac{C_1}{\lambda} \cosh \lambda x + \frac{C_2}{\lambda} \sinh \lambda x + C_3 - \frac{1}{2GI_T}(2m_0 + m_1 \frac{x}{l})x$$ (10)

$$\phi'' = C_1 \sinh \lambda x + C_2 \cosh \lambda x - \frac{1}{GI_T}(m_0 + m_1 \frac{x}{l})$$ (11)

$$\phi''' = C_1 \lambda \cosh \lambda x + C_2 \lambda \sinh \lambda x - \frac{m_1}{GI_T l}$$ (12)

The values of the constants C_1 to C_4 can be derived with respect to the kinematic boundary conditions of

the problem. For the case of warping-free sections, where C_M = 0, the differential equation is shortened, leading to the St. Venant torsion problem.

3 Model and Results

The properties of the analysed model, are defined in Table 1. The corresponding results are presented in Table 2. Figure 4 shows the deformed shape of the structure and the angle of twist.

Table 1: Model Properties

Material Properties	Cross-sectional Properties	Loading
$E = 217396.3331684\,N/mm^2$	$l = 1\,m$	$m_T = 1\,Nmm/mm$
$G = 81386.6878\,N/mm^2$	$h = 80\,mm$	
$\nu = 0.33557673$	$t = 2\,mm$	
	$b = 40\,mm$	
	$C_M = 0.323 \times 10^8\,mm^6$	
	$I_T = 431.979\,mm^4$	

Table 2: Results

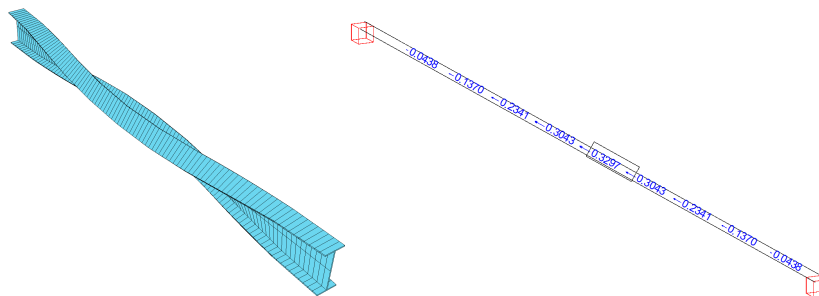|	Twist in x-direction	Ref. [1]
$\phi_x\,[mrad]$	0.329659	0.329262

Figure 4: Deformed Stucture

4 Conclusion

This example presents the warping torsion problem. The total torsional moment resisted by the cross-section is the sum of that due to pure torsion, which is always present, and that due to warping. It has been shown that the behaviour of the beam for warping is captured correctly.

5 Literature

[1] C-N. Chen. "The Warping Torsion of a Bar Model of the Differential Quadrature Element Method". In: *Computers and Structures* 66.2-3 (1998), pp. 249–257.

[2] S. Timoshenko. *Strength of Materials, Part II, Advanced Theory and Problems*. 2nd. D. Van Nostrand Co., Inc., 1940.

[3] F.P. Beer, E.R. Johnston, and J.T. DeWolf. *Mechanics of Materials*. 4th. McGraw-Hill, 2006.

[4] P. Seaburg and C.J. Carter. *Steel Design Guide Series 9: Torsional Analysis of Structural Steel Members*. AISC. 2003.

[5] C. Petersen. *Stahlbau*. 2nd. Vieweg, 1990.